

Wednesday, August 27:

• Syllabus & Course outline.

← Office Hours, GenAI statement,
Discord, Gradescope, no recordings

• Series and Linear Algebra (with a little optimization)

• Complex Analysis

• Integral Transforms and Fourier Series.

Series Refresher

Let's recall what a sequence is first.

Def: Sequence

A sequence $\{a_n: n=1, 2, 3, \dots\}$ is an ordered list of objects a_n .

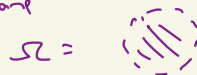
For us: we care about ordered lists of

• Numbers, real ($a_n \in \mathbb{R}$) and complex ($z_n \in \mathbb{C}$)

later [• Vectors, real ($\vec{a}_n \in \mathbb{R}^k$) and complex ($\vec{z}_n \in \mathbb{C}^k$)

• Functions, real and complex $f: S \rightarrow \mathbb{R} \rightarrow \mathbb{C}$

↑
some set, like a region
in the plane



Examples:

• $0, 1, 0, 1, 0, 1, \dots$

• $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

• $1, 1+x, 1+x+\frac{x^2}{2!}, 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}, \dots$

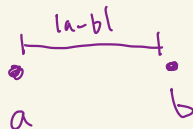
we would like to know when a sequence is "going somewhere".

This is the notion of limits and convergence.

(Note that we need a way to measure distance to describe this!)

For real numbers a and b , the distance between them is given by

$$\text{dist}(a, b) = |a - b|$$



• Later: notions of distance for vectors and functions.

Def: Convergence and limits

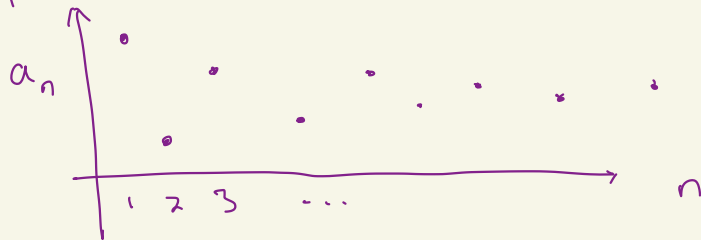
We say a sequence a_n of real numbers converges to a limit a ,

$$\lim_{n \rightarrow \infty} a_n = a$$

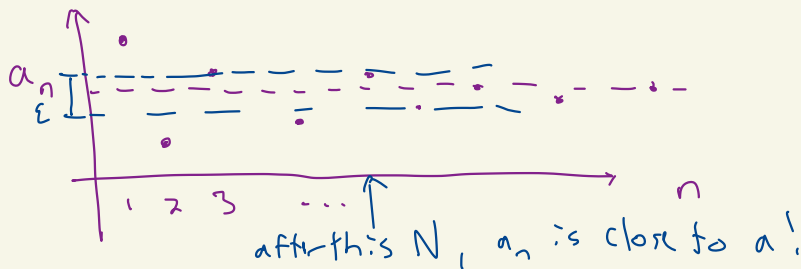
if for every distance $\varepsilon > 0$, we can find a large enough N such that the tail of the sequence has

$$|a_n - a| < \varepsilon \quad \text{for every } n \geq N$$

Picture

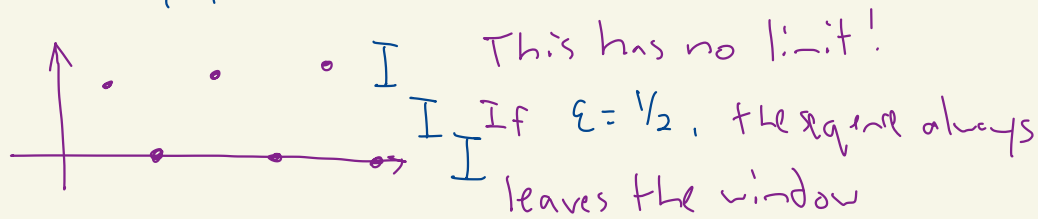


If distance $\varepsilon = \dots$



Many sequences do not converge:

Ex: 0, 1, 0, 1, 0, 1



Any sequence that does not converge is said to diverge.
A series is just a special type of sequence.

Def. Partial sums, Series

Let a_k be a sequence, and let

$$S_N = \sum_{k=1}^N a_k$$

← $S_N = N^{\text{th}}$ partial sum.

- We call S_N a series. We say the infinite series $\sum a_k$ converges if $\lim_{N \rightarrow \infty} S_N = S$, or

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N a_k = S \leftarrow \text{finite real number.}$$

we often write $S = \sum_{k=1}^{\infty} a_k$.

- We say the series $\sum a_k$ converges absolutely if the series $\sum |a_k|$ converges.

Prime Example: Geometric Series

what is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$?

Well, let $r = \frac{1}{2}$,

$$S_N(r) = \sum_{k=0}^N r^k = 1 + r + \dots + r^N$$

Notice that

$$(1-r)(1+r+r^2+\dots+r^N) = \begin{array}{l} 1+r+r^2+\dots+r^N \\ -r-r^2-\dots-r^N-r^{N+1} \\ \hline 1-r^{N+1} \end{array}$$

$$= 1 - r^{N+1}$$

$$\Rightarrow S_N(r) = \frac{1-r^{N+1}}{1-r} \quad (\text{as long as } r \neq 1)$$

When does this sequence converge?

• If $r > 1$,

$$\lim_{N \rightarrow \infty} S_N(r) = \frac{1-r^{N+1}}{1-r} = \frac{1-\infty}{1-r} = -\infty$$

• If $-1 < r < 1$,

$$\lim_{N \rightarrow \infty} S_N(r) = \frac{1-r^{N+1}}{1-r} = \frac{1-0}{1-r} = \frac{1}{1-r}$$

• If $r < -1$,

$$\lim_{N \rightarrow \infty} S_N(r) = \frac{1-r^{N+1}}{1-r} = \dots$$

Divergent!

Theorem:

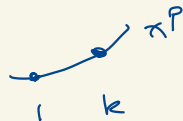
Let $-1 < r < 1$. Then the geometric series converges absolutely, and

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Example: p-series

Consider $\sum_{k=1}^{\infty} k^p$. Does this series converge?

- If $p > 0$, this is hopelessly divergent:
since $1 \leq k$, and the function x^p is increasing,
 $1^p \leq k^p$, and so



$$1 + 1 + \dots = \sum_{k=1}^{\infty} 1 \leq \sum_{k=1}^{\infty} k^p$$

$\uparrow = \infty$, diverges $\Rightarrow$ sum diverges